

Polynomial method in coding and information theory

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Abstract

Polynomial, or Delsarte's, method in coding theory accounts for a variety of structural results on, and bounds on the size of, extremal configurations (codes and designs) in various metric spaces. In recent works of the authors the applicability of the method was extended to cover a wider range of problems in coding and information theory. In this paper we present a general framework for the method which includes previous results as particular cases. We explain how this generalization leads to new asymptotic bounds on the performance of codes in binary-input memoryless channels and the Gaussian channel, which improve the results of Shannon et al. of 1959-67, and to a number of other results in combinatorial coding theory.

1 Introduction: Some problems of coding and information theory

Let X be a metric space with distance function $\partial(\cdot, \cdot)$. A *code* C is an arbitrary finite subset of X . The number $d(C) = \min_{\mathbf{c}_1, \mathbf{c}_2 \in C, \mathbf{c}_1 \neq \mathbf{c}_2} \partial(\mathbf{c}_1, \mathbf{c}_2)$ is called the *distance* of C . The study of codes was initiated in the context of transmission of information over noisy channels [37].

The motivating example is codes in the binary Hamming space $H^n = \mathbb{F}_2^n$ with the metric $\partial(\mathbf{x}, \mathbf{y}) = |\{e \in \{1, 2, \dots, n\} \mid x_e \neq y_e\}|$. This example corresponds to transmission over the binary symmetric channel (BSC). Suppose that a vector \mathbf{x} is transmitted. In the channel each coordinate is inverted with probability p and left intact with probability $1 - p$ and different coordinates are subjected to the error process independently. Let $P(\mathbf{y}|\mathbf{x})$ be the conditional probability distribution induced on H^n by this channel. $P(\mathbf{y}|\mathbf{x})$ is a monotone (decreasing) function of the distance $\partial(\mathbf{x}, \mathbf{y})$; hence it is possible to study the performance of codes in geometric terms. In particular, for small p the most important parameter of the code is its distance. This gives an information-theoretic reason to look for codes of a given size with large distance. There are also other combinatorial and geometric reasons for this interest; we outline them below.

Now suppose that $\mathbf{x} \in \mathbb{R}^n$ and the error process in the channel is described as follows: for a transmitted vector \mathbf{x} we receive from the channel a vector $\mathbf{y} = \mathbf{x} + \mathbf{e}$, where each coordinate of \mathbf{e} is a Gaussian $(0, \sigma^2)$ random variable. A consistent definition of capacity of such a channel is obtained if the input signals satisfy some sort of energy constraints. Typically one assumes that the energy, or the average energy of input signals does not exceed a given number $A\sigma^2$ per dimension, where A is a positive number called the “signal-to-noise ratio.” Shannon [38] has shown that for a set of input signals of sufficiently

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large size the study of the channel is reduced to considering signals of *constant* energy equal to $\sigma\sqrt{An}$, that is, points on the n -dimensional sphere. Thus our second example will be $X = S^{n-1}(\mathbb{R})$, the unit sphere in \mathbb{R}^n , with the Euclidean distance $\partial(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^{1/2}$.

The third standard example is $X = \{\mathbf{x} \in H^n \mid \#\{i : x_i = 1\} = v\}$, called the binary Johnson space $J^{n,v}$. Since this space is a subset of H^n , it is again associated with transmission over the BSC. However, the interest in the Johnson space is largely determined by the fact that combinatorially it can be studied by methods similar to the Hamming case [14], [8] and has strong connections to the latter [34], [36].

The theory and a part of results outlined in this paper are valid in a large class of finite spaces that afford the structure of an association scheme, and in the infinite case, in compact two-point homogeneous spaces. However below we concentrate on the above examples since they give rise to central asymptotic problems of coding and information theory that can be treated in geometric terms. Let us outline these problems. We use the mixed entropy and entropy functions

$$\begin{aligned} T_s(x, y) &= x \log_q(q-1) - x \log_s y - (1-x) \log_s(1-y), \\ H_s(x) &= T_s(x, x), \end{aligned}$$

and omit the subscript if the logarithms are taken base e .

1.1 The size-distance (R - δ) problem.

Let $A(X; n, d) = \max_{C \in X} \{|C| \mid d(C) = d\}$ be the maximal possible size of the code with a given distance. Finding this function is one of the central problems of coding theory. Apart from a number of particular cases for small n this problem is unsolved.

First let $X = H^n$. Let $R = (1/n) \log_2 |C|$ and $\delta = d(C)/n$ be the *rate* and the *relative distance* of the code. Clearly, $0 \leq R \leq 1, 0 \leq \delta \leq 1$. Let

$$\bar{R}(\delta) = \limsup_{n \rightarrow \infty} \frac{\log_2 A(H^n; n, d)}{n}, \quad \underline{R}(\delta) = \liminf_{n \rightarrow \infty} \frac{\log_2 A(H^n; n, d)}{n},$$

where the limits are computed over all sequences of codes C_n for which $\limsup_{n \rightarrow \infty} \frac{d(C_n)}{n} \geq \delta$. Below we assume that these two functions have a common limit, denoted $R(\delta)$ (if they do not, the upper bounds become bounds on \bar{R} and the lower ones on \underline{R}). It is clear that $R(\delta)$ is a monotone decreasing function of δ ; its inverse is denoted below by $\delta(R)$.

It is known and easily proved that $R(\delta) = 0$ for $\delta \in [\frac{1}{2}, 1]$. Otherwise the best known bounds on $\delta(R)$ have the form:

$$\delta(R) \geq \delta^{(\text{vg})}(R) := H_2^{-1}(1-R) \quad [22], [43] \quad (1)$$

$$\delta(R) \leq \delta^{(\text{lp})}(R) := \min_{\substack{0 \leq \beta \leq \alpha \leq 1/2 \\ H_2(\alpha) - H_2(\beta) = 1-R}} 2 \frac{\alpha(1-\alpha) - \beta(1-\beta)}{1 + 2\sqrt{\beta(1-\beta)}}. \quad [34] \quad (2)$$

Let $X = S^{n-1}(\mathbb{R})$. In this case the corresponding functions are written in the form $R(d), d(R)$, where $d(R), 0 \leq d(R) \leq 2$, is the limit value of the Euclidean distance of codes of rate $R := \frac{1}{n} \log |C|$. We have

$$d(R) \geq d^{(\text{s})}(R) := \sqrt{2(1 - \sqrt{1 - e^{-2R}})} \quad (0 \leq R < \infty) \quad [38] \quad (3)$$

$$d(R) \leq d^{(\text{kl})}(R) := \frac{\sqrt{2}(\sqrt{1+\rho} - \sqrt{\rho})}{\sqrt{1+2\rho}} \quad [24],$$

where in the last formula ρ is the root of

$$R = (1 + \rho)H\left(\frac{\rho}{1 + \rho}\right) \quad (0 \leq R < \infty). \quad (4)$$

Lower bounds (1) and (3) are obtained by random choice. The upper bounds are obtained by the polynomial method which is outlined in the next section.

1.2 Error probability of decoding.

Though the packing (R - δ) problem has received more attention, Shannon's original motivation was reliable transmission of information over channels. Let $X = H^n$ and let $C \subset X$ be a code used for transmission over the binary symmetric channel. A *decoding* is a (partial) mapping $\psi : X \rightarrow C$. Let $S(t, \mathbf{c})$ be a sphere of radius t around a point \mathbf{c} . Consider the decoding defined on $\cup_{\mathbf{c} \in C} S(t, \mathbf{c})$ as follows:

$$\psi_t(\mathbf{y}) = \mathbf{c} \quad \text{if } \mathbf{y} \in S(t, \mathbf{c}) \text{ and } \partial(\mathbf{y}, \mathbf{c}) \leq \partial(\mathbf{y}, \mathbf{c}') \text{ for all } \mathbf{c}' \in C.$$

Clearly, if $t \leq \lfloor (d(C) - 1)/2 \rfloor$, the decoding result is defined uniquely; otherwise we agree that ties are broken arbitrarily. As long as $t \leq \lfloor (d(C) - 1)/2 \rfloor$, the spheres $S(t, \mathbf{c})$ are disjoint; for larger t some of them intersect. Starting with a certain value of t (called the *covering radius* of C) their union covers the entire X . In this case the decoding is called *complete*. Let $P_{de}(H^n; C, p)$ be the average error probability of complete decoding for a code C used over the BSC with error probability p :

$$P_{de}(H^n; C, p) := \frac{1}{|C|} \sum_{\mathbf{x} \in C} P_{de}(\mathbf{x}),$$

where the last probability describes the event that the transmitted code vector is \mathbf{x} and the decoding result is a code vector $\mathbf{x}' \neq \mathbf{x}$. Let

$$P_{de}(H^n; n, R, p) = \max P_{de}(H^n; C, p),$$

where the maximum is taken over all codes of rate $\geq R$.

Obviously, these definitions are valid for any metric space; in particular for $S^{n-1}(\mathbb{R})$ and the Gaussian channel. Therefore, we also consider the error probability of decoding $P_{de}(S^{n-1}; n, R, A)$, defined analogously. It is known that $P_{de}(X; n, R)$ falls exponentially for both $X = H^n$ [17] and $X = S^{n-1}$ [38]; consider therefore the exponents

$$\begin{aligned} \bar{E}_{de}(H^n; R, p) &= \limsup_{n \rightarrow \infty} -\frac{1}{n} \log_2 P_{de}(H^n; R, p, n) \\ \bar{E}_{de}(S^{n-1}; R, A) &= \limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_{de}(S^{n-1}; R, A, n). \end{aligned}$$

After Shannon [38] the best attainable error exponent is called the *reliability function* of the channel. Computing the reliability function of these and other channels dominated information theory through the end of the 1960s [20]. Even in the simplest cases mentioned this problem is still unsolved. Upper bounds on $\bar{E}_{de}(H^n; R, p)$ were derived in [39]; see also [33]. Lower bounds on $\bar{E}_{de}(H^n; R, p)$ were given in [17], [19].¹ Lower and upper bounds on $\bar{E}_{de}(S^{n-1}(\mathbb{R}); R, p)$ were obtained in [38]; see also [24].

For $X = H^n$ coding theorists have also studied the other limiting case of decoding, that of decoding radius $t = 0$, called *error detection*. The probability of undetected error is defined analogously:

$$P_{ue}(H^n; C, p) := \frac{1}{|C|} \sum_{\mathbf{x} \in C} P_{ue}(\mathbf{x}),$$

¹Lower bounds on the reliability function constitute Shannon's channel coding theorem, proved in the general case by Feinstein [18], see also Khinchin [26].

where the probability $P_{ue}(\mathbf{x})$ corresponds to the event that the received vector, equal to $\mathbf{x} + \mathbf{e}$, $\mathbf{e} \in H^n \setminus \mathbf{0}$, is itself in C .

2 Polynomial method

Delsarte [13] suggested a method of deriving upper bounds on the size of a code with a given distance by optimizing a certain functional on the cone of polynomials of degree at most n . The formalism of the method can be developed either in the context of association schemes [14], [8] or of harmonic analysis on noncommutative compact groups [24], [29].

We again begin with the binary Hamming space H^n . The main role in the method is played by the distance distribution of codes. Let $C \subset H^n$ be a code. Its (average) *distance distribution* is given by $\mathcal{A} = (\mathcal{A}_i, 0 \leq i \leq n)$, where

$$\mathcal{A}_i = \frac{1}{|C|} |\{(\mathbf{c}, \mathbf{c}') \in C^2 \mid \partial(\mathbf{c}, \mathbf{c}') = i\}|.$$

Let $X = H^n$ and let $(K_k(x), k = 0, 1, \dots)$ be the system of Krawtchouk polynomials [41], i.e., polynomials orthogonal on the set $(0, 1, 2, \dots, n)$ with weight $\mu(i) = \binom{n}{i} 2^{-i}$. Let C be a code with the distance distribution \mathcal{A} . The *MacWilliams transform* of \mathcal{A} is a vector $\mathcal{A}' = \frac{1}{|C|} \mathbf{A} \mathbf{K}$, where $\mathbf{K} = (K_k(i), 0 \leq i, k \leq n)$ is the $(n+1) \times (n+1)$ Krawtchouk matrix with rows numbered by i and columns by k . One of the central properties of codes, the *Delsarte inequalities* [13], [14], is that the components of \mathcal{A}' are nonnegative.

Now let $f(x) = \sum_{i=0}^n f_i K_i(x)$ be a polynomial of degree at most n written in the Krawtchouk basis. The definition of the MacWilliams transform implies the following useful identity [13]:

$$|C| \sum_{i=0}^n f_i \mathcal{A}'_i = \sum_{i=0}^n f(i) \mathcal{A}_i. \quad (5)$$

The following theorem, proved in particular cases in [13], [1], [32], [4], is the main general result of this paper. It accounts for the new upper bounds on the reliability functions of the next sections as well as for some other estimates of code parameters.

Theorem 1 *Let C be a code with distance distribution \mathcal{A} . Let $f(x) = \sum_{k=0}^n f_k K_k(x)$, $f_k \geq 0$, $1 \leq k \leq n$, be a polynomial of degree at most n . Let $F = \sum_{i=1}^n g(i) \mathcal{A}_i$ be a function on C and suppose that $f(i) \leq g(i)$, $0 \leq i \leq n$. Then*

$$F \geq |C| f_0 - f(0).$$

PROOF. By the inequality $\mathcal{A}' \geq 0$ and (5) we obtain

$$|C| f_0 \leq |C| \sum_{i=0}^n f_i \mathcal{A}'_i = f(0) + \sum_{j=1}^n f(j) \mathcal{A}_j \leq f(0) + \sum_{j=1}^n g(j) \mathcal{A}_j = f(0) + F.$$

where we have used the fact that $\mathcal{A}'_0 = 1$. □

Examples.

1. Probability of undetected error. Let $0 \leq p \leq \frac{1}{2}$ and $g(i) = p^i (1-p)^{n-i}$. Then $F = P_{ue}(H^n; C, p)$.
2. Delsarte's linear programming bound. Let $g(i) = 0$, $1 \leq i \leq n$, then $F = 0$. Suppose that the code C in Theorem 1 has distance d . Then $\mathcal{A}_i = 0$ for $i = 1, \dots, d-1$. Hence it suffices to assume

that $f(i) \leq 0$ for $i = d, d+1, \dots, n$. Assuming in addition that $f_0 > 0$, we obtain Delsarte's linear programming bound on the size of a code with distance d [13]:

$$|C| \leq \inf_f \left\{ \frac{f(0)}{f_0} \mid f_0 > 0, f_k \geq 0, 1 \leq k \leq n; f(i) \leq 0, i = d, d+1, \dots, n \right\}. \quad (6)$$

The problem of finding stationary points of the functional $f(0)/f_0$ has been one of the central in combinatorial coding theory since 1972 (see [31]).

3. Let $1 \leq w \leq n$ be an integer and let $g(i) = \binom{n-i}{n-w}$. We obtain a set of code invariants $F_w = \sum_{i=0}^w \binom{n-i}{n-w} \mathcal{A}_i$. The numbers F_w (binomial moments of the distance distribution) are related to numerous combinatorial invariants [1], [10], [9], for instance, the cumulative size of subcodes of restricted support, and, in the linear case, to the higher weight enumerators, rank polynomial, Tutte polynomial, etc.

4. Suppose that in Theorem 1 $f(i) > 0$ for $0 \leq i \leq w$ and $f(i) \leq 0$ for $w+1 \leq i \leq n$, where $w \in [1, n]$ is a parameter. Put $g(i) = f(i)$. Then the theorem implies in a way similar to Example 2 the inequality

$$\sum_{i=1}^w f(i) \mathcal{A}_i \geq |C| f_0 - f(0).$$

In other words, there exists a number $j, 1 \leq j \leq w$, such that

$$\mathcal{A}_j \geq \frac{|C| f_0 - f(0)}{f(j)}. \quad (7)$$

This is one of the main results in [32]. Since the polynomial f has to satisfy the same conditions as in Example 2, it is possible to use the known results in the R - δ problem to derive specific lower bounds on the distance distribution of codes [32] (see Sect.4).

Let $C \subset J^{n,v}$ be a code in the Johnson space and $\mathcal{A} = (\mathcal{A}_{2i}, 0 \leq i \leq v)$ be its average distance distribution. Let $Q_k^v(x)$ be a family of Hahn polynomials orthogonal on the set $(0, 1, \dots, v)$ with weight $\mu(i) = \frac{\binom{v}{i} \binom{n-v}{i}}{\binom{n}{i}}$. Then as above one can consider the transformed distribution $\mathcal{A}' = \frac{1}{|C|} \mathcal{A} \mathbf{Q}$, where $\mathbf{Q} = (Q_k^v(i), 0 \leq i, k \leq v)$ is the Hahn matrix. Again by Delsarte's theory [14] the components of \mathcal{A}' are nonnegative. Thus, one can consider the invariants of Examples 1-4 for the Johnson space. In particular, inequalities (6),(7) are straightforward [14],[32], where this time $f(x)$ is a polynomial with positive Hahn-Fourier coefficients.

One particular reason to study bounds of the form (6), (7) in the Johnson space follows from the fact that one can translate them to the Hamming space. Namely, since $J^{n,v} \subset H^n$, an easy averaging argument (the *multiple packing principle* [11]) shows that the maximum sizes of codes in H^n and $J^{n,v}$ are related as follows:

$$\binom{n}{v} A(H^n; n, d) \leq 2^n A(J^{n,v}; n, d). \quad (8)$$

Thus, upper bounds on $A(J^{n,v}; n, d)$ also give upper bounds on codes in the Hamming space (an important example being (2)). Several generalizations of this argument are known; in particular, one can prove a lower bound of the form (7) in $J^{n,v}$ and then translate it to H^n [32]. An alternative approach to (8) is based on the fact that positive definite functions in the Hahn basis are also positive definite in the Krawtchouk basis [36]. This gives an analytic method of deriving inequalities of the type (8), which is useful for those code invariants that are not well defined in the Johnson space, hence do not carry a geometric meaning. For instance, this is the case with the F_w -invariants of Example 3 [1].

Finally, consider the case $C \subset S^{n-1}(\mathbb{R})$. Let $\partial(\mathbf{c}, \mathbf{c}') = \|\mathbf{c} - \mathbf{c}'\|$ be the Euclidean distance in \mathbb{R}^n . It is convenient to define the distance distribution of C with the help of the function $t(x) = 1 - \frac{x^2}{2}$. In particular, $t(\partial(\mathbf{c}, \mathbf{c}')) = \langle \mathbf{c}, \mathbf{c}' \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^n . Let

$$a(s, t) := \frac{1}{|C|} |\{(\mathbf{c}, \mathbf{c}') \in C^2 : s \leq \langle \mathbf{c}, \mathbf{c}' \rangle \leq t\}|$$

be the *distance density* of C . Delsarte's inequalities in this case take on the form [24]

$$\int_{-1}^1 \delta(t) a(t, t) P_k^{\lambda, \lambda}(t) dt = \sum_{\mathbf{c}, \mathbf{c}' \in C} P_k^{\lambda, \lambda}(\langle \mathbf{c}, \mathbf{c}' \rangle) \geq 0, \quad k = 0, 1, \dots,$$

where $P_k^{\alpha, \beta}(x)$ is the Jacobi polynomial, $\delta(t)$ is the delta-function, and $\lambda = (n-3)/2$. The analog of (7) in this case is given by the following theorem.

Theorem 2 [4] *Let $C \subset S^{n-1}(\mathbb{R})$ be a code and let m be a integer. Let $-1 \leq u_0 < t(d(C))$ and suppose that $u_0 < u_1 < \dots < u_{m-1} < u_m = t(d(C)) < 1$ are the defining points of a partition of the segment $[u_0, t(d(C))]$ into m equal segments $U_i = [u_i, u_{i+1}]$.*

Suppose that $f(x) = \sum_{k=0}^l f_k P_k^{\alpha, \alpha}(x)$ is a polynomial of degree l such that $f_k \geq 0, 1 \leq k \leq l$, and $f(x) \leq 0, -1 \leq x \leq u_0$, $f(x) \geq 0, u_0 \leq x \leq 1$. Then there exists a number $i, 0 \leq i \leq m-1$, and a point $s \in U_i$ such that

$$a(u_i, u_{i+1}) \geq \frac{f_0 |C| - f(1)}{m f(s)}. \quad (9)$$

The three metric spaces considered above (and many other spaces) can be studied from one and the same point of view. This is the principal achievement of [24]. It turns out that the polynomials associated with the space $(K_k, H_k, P_k^{\alpha, \beta})$ represent the zonal spherical kernels that arise in the analysis of irreducible unitary representations of the isometry group of the space. Spaces in which zonal spherical functions are expressed by univariate polynomials are sometimes called *polynomial* [29], [23].

3 Asymptotics of orthogonal polynomials

To derive asymptotic bounds on the distance distribution of codes and other invariants we need asymptotic formulas for orthogonal polynomials involved in inequalities (7), (9). These problems have been studied more or less independently in coding theory [34], [24], [29], [25], [32], [1], [4] and analysis [35], [12], [21], [16], [15], [28]. We quote results from the coding-theory side since they are in the form better suited to our needs.

Asymptotics of extremal zeros found in [34], [24] were used in these papers to derive the bounds $\delta^{(\text{lp})}(R)$ and $d^{(\text{kl})}(R)$, respectively. However, to derive bounds on code invariants we need to find the behavior of the polynomials from the extremal zero to the end of the orthogonality segment.

Krawtchouk polynomials. $K_k(0) = \binom{n}{k}$ and the polynomial is monotone decreasing in the segment $[0, x_1(K_k)]$, where x_1 is the smallest zero of $K_k(x)$. Let $k/n \rightarrow \tau$ as $n \rightarrow \infty$. It is known [34], [29] that $x_1(K_k) \approx \frac{n}{2} - \sqrt{k(1-k)}$. An asymptotic expression for the exponent of $K_k(x)$ for $x \in [0, x_1(K_k)]$ was derived in [25]. It has the form

$$\frac{1}{n} \log_2 K_k(\xi n) = H_2(\tau) + \int_0^\xi \log_2 \frac{1 - 2\tau + \sqrt{(1-2\tau)^2 - 4y(1-y)}}{2-2y} dy + o(1). \quad (10)$$

Hahn polynomials. The smallest zero of $Q_k^v(x)$ behaves as follows [34], [29]:

$$x_1(Q_k^v) \approx \frac{v(n-v) - k(n-k)}{n + 2\sqrt{k(n-k)}}.$$

Similarly to [25] we have [1],[32]

$$\begin{aligned} \frac{1}{n} \log_2 Q_k^v(\xi n) = & H_2(\beta) + \int_0^\xi \log_2 \left[\frac{\alpha(1-\alpha) - y(1-2y) - \beta(1-\beta)}{2(\alpha-y)(1-\alpha-y)} \right. \\ & \left. + \frac{\sqrt{[\alpha(1-\alpha) - y(1-2y) - \beta(1-\beta)]^2 - 4(\alpha-y)(1-\alpha-y)y^2}}{2(\alpha-y)(1-\alpha-y)} \right] dy + o(1), \end{aligned} \quad (11)$$

where $n \rightarrow \infty, v = \alpha n, k = \beta n, x \in [0, x_1(Q_k^v)]$.

Jacobi polynomials. The asymptotic expression for the largest zero of $P_k^{ak,bk}$ has the form [24], [35]

$$x_1^{a,b} := x_1(P_k^{ak,bk}) \approx \frac{4\sqrt{(a+b+1)(a+1)(b+1)} - a^2 - b^2}{(a+b+2)^2}.$$

The smallest zero then is $-x_1^{b,a}$. The asymptotic behavior of the exponent of $P_k^{ak,bk}, k \rightarrow \infty$, in the entire orthogonality segment was found in [4]. We quote one of the results in [4]: let $x \in [-1, -x_1^{b,a} - \epsilon_k] \cup [1, x_1^{a,b} + \epsilon_k]$, where $\epsilon_k = k^{-\gamma}, 0 \leq \gamma \leq 1/2$. Then

$$\begin{aligned} \frac{1}{k} \ln |P_k^{\alpha,\beta}(x)| = & (1+a)H\left(\frac{a}{1+a}\right) \\ & \mp \int_x^1 \frac{(a + (a+b)z - b) \mp \sqrt{(a + (a+b)z - b)^2 - 4(1-z^2)(1+a+b)}}{2(1-z^2)} dz + o(1), \end{aligned} \quad (12)$$

where the $-$ sign corresponds to the left of the 2 segments in the domain of x and the $+$ to the right of them.

4 Lower bounds on code invariants

Specifications of Theorem 1 enable one to prove a large number of results on code properties. In this section we present estimates on the distance distribution of codes and related invariants.

Let $C \subset H^n$ be a code of rate R and \mathcal{A} its distance distribution. Theorem 1 together with (10) implies the following

Theorem 3 [32] *For any code of sufficiently large length n there exists a number $\xi \in [0, 1/2 - \sqrt{\tau(1-\tau)}], 0 \leq \tau \leq H_2^{-1}(R)$, such that*

$$\mathcal{A}_{\lfloor \xi n \rfloor} \geq R - H_2(\tau) - 2I(\xi, \tau) - o(1),$$

where $I(\xi, \tau)$ is the integral on the right-hand side of (10).

The bound in this theorem can be slightly improved with the help of a generalization of (8) and asymptotics (11) [32].

The lower estimate on F_w invariants of C , which is also proved with the help of Theorem 1, has the following form.

Theorem 4 [1] *For any code of sufficiently large length n*

$$\frac{1}{n} \log_2 F_{[2\omega n]} \geq R - 1 + H_2(\omega^*) + (1 - \omega^*)H_2\left(\frac{1 - 2\omega}{1 - \omega^*}\right) - o(1),$$

where

$$\omega^* = \begin{cases} \omega, & \delta^{(\text{lp})}(R) \leq \omega \leq 1 \\ \delta^{(\text{lp})}(R), & \delta^{(\text{lp})}(R)/2 \leq \omega \leq \delta^{(\text{lp})}(R), \end{cases}$$

and $\delta^{(\text{lp})}(\cdot)$ is defined in (2).

For $X = S^{n-1}(\mathbb{R})$ Theorem 2 and (12) imply the following

Theorem 5 [4] *Let C be a code of rate R . Let $\gamma \in [0, \rho]$, where ρ is the root of (4) be a fixed number.*

Then there exists a number x , $\frac{2\sqrt{\gamma(1+\gamma)}}{1+2\gamma} \leq x \leq 1$, such that for sufficiently large n

$$\frac{1}{n} \ln a(x, x + \frac{1}{n}) \geq 4\gamma(1+\gamma) \int_x^1 \frac{dz}{z + \sqrt{z^2 - 4(1-z^2)\gamma(1+\gamma)}} - (1+\gamma)H\left(\frac{\gamma}{1+\gamma}\right) + R - o(1).$$

These estimates lead to new upper bounds on the reliability function of the BSC [32], the Gaussian channel [4], of the exponent of error detection [1], [4] (see the next section) and on a number of other parameters of codes. The approach developed in [24] enables us to derive similar bounds on the distance distribution of codes in projective real and complex spaces [4].

5 Reliability functions and error detection

The key to the results of this section is given by the following observation: if a code vector that has many close neighbors is sent over the channel, the error probability of decoding cannot be too low. Together with the estimates of the previous section and some other combinatorial and geometric considerations this leads to the following results.

Theorem 6 [32] *The reliability function of BSC with error probability p satisfies the upper bound*

$$\bar{E}_{de}(H^n; R, p) \leq \max_{\alpha, \beta, \xi, \delta} E_{\alpha, \beta, \xi, \delta},$$

where

$$E_{\alpha, \beta, \xi, \delta} = \min \left(-\delta \log_2 \sqrt{4p(1-p)}, -\tilde{\nu} - \xi \log_2 \sqrt{4p(1-p)} \right),$$

and α, β, δ , and ξ are such that $0 \leq \beta \leq \alpha \leq 1/2$, $H_2(\alpha) - H_2(\beta) = 1 - R$, $\delta \in [0, \delta^{(\text{lp})}(R)]$,

$$\xi \in \left[0, 2 \frac{\alpha(1-\alpha) + \beta(1-\beta)}{1 + 2\sqrt{\beta(1-\beta)}} \right];$$

$$\begin{aligned} \tilde{\nu} = \min \left(\nu, \xi + (1-\xi)H_2(p) - \max_{\eta \in [\delta p/2, \min(\delta/4, p(1-\xi))]} \left(\delta H_2\left(\frac{2\eta}{\delta}\right) \right. \right. \\ \left. \left. + (\xi - \delta/2)H_2\left(\frac{\xi - 2\eta}{2\xi - \delta}\right) + (1 - \xi - \delta/2)H_2\left(\frac{p(1-\xi) - \eta}{1 - \xi - \delta/2}\right) \right) \right), \end{aligned}$$

$$\nu = R - 1 + H_2(\beta) + 2H_2(\alpha) - 2q(\alpha, \beta, \xi/2) - \xi - (1-\xi)H_2\left(\frac{\alpha - \xi/2}{1 - \xi}\right),$$

and $q(\alpha, \beta, \xi)$ is the function on the right-hand side of (11).

As shown in [39], given any convex upper bound on $E(H^n; R, p)$ one can draw a common tangent to it and the *sphere-packing bound* (one of the bounds in [39]), and the segment between the tangency points will also give an upper bound on $E(H^n; R, p)$, the so-called *straight-line bound*. Together with the last theorem this gives the best upper bound to-date on $E(H^n; R, p)$. Standard methods of information theory [20] enable one to extend this result to memoryless channels with binary input alphabet.

Error-correcting properties of codes on $S^{n-1}(\mathbb{R})$ are given by the following theorem, whose proof relies, in particular, on Theorem 2 and (12).

Theorem 7 [4] *The reliability function of the Gaussian channel with signal-to-noise ratio A satisfies the upper bound*

$$\bar{E}_{de}(S^{n-1}; R, A) \leq \min_{0 \leq \gamma \leq \rho} \max_{w, d} \left[\min\left(A \frac{d^2}{8}, A \frac{w^2}{8} - \mathcal{L}(w, d, \gamma)\right) \right], \quad (13)$$

where

$$0 \leq d \leq \frac{\sqrt{2}(\sqrt{1+\rho} - \sqrt{\rho})}{\sqrt{1+2\rho}}, \quad d \leq w \leq \frac{\sqrt{2}(\sqrt{1+\gamma} - \sqrt{\gamma})}{\sqrt{1+2\gamma}},$$

ρ is the root of (4)

$$\mathcal{L}(w, d, \gamma) = \min \left\{ \frac{Ad^2w^2}{8(4w^2 - d^2)}, F\left(1 - \frac{1}{2}w^2, \gamma\right) \right\},$$

$$F(x, \gamma) = R - (1 + \gamma)H\left(\frac{\gamma}{1 + \gamma}\right) + \int_x^1 \frac{4\gamma(1 + \gamma)dz}{z + \sqrt{z^2 - 4(1 - z^2)\gamma(1 + \gamma)}}.$$

The remark made after Theorem 6 regarding the straight-line bound is valid for the Gaussian channel as well; taken together, this, again, is the best result known to-date.

Results of the previous sections also lead to the following upper bound on the exponent of error detection $\bar{E}_{ue}(H^n; R, p)$ for codes on the BSC with crossover probability p .

Theorem 8

$$\bar{E}_{ue}(H^n; R, p) \leq \begin{cases} 1 - R - H_2(\delta^{(\text{lp})}(R)) + T_2(\delta^{(\text{lp})}(R), p), & 0 \leq R \leq R^{(\text{lp})}(p) \\ 1 - R, & R^{(\text{lp})}(p) \leq R \leq 1, \end{cases}$$

where $\delta^{(\text{lp})}(\cdot)$ is given by (2) and $R^{(\text{lp})}(\cdot)$ is its inverse function.

This theorem was proved in [1] via lower bounds on the F_w -invariants (Theorem 4) and in [32] with the use of Theorem 3. Together with known lower bounds (see, e.g., [30]), it shows that the function $E_{ue}(H^n; R, p)$ is known exactly for $R \in [R^{(\text{lp})}(p), 1]$.

It is worth mentioning that if the Varshamov-Gilbert bound (1) is tight (as is widely believed) then the known lower bounds on $E_{de}(H^n; R, p)$ and $E_{ue}(H^n; R, p)$ are also tight. The same is true with respect to the Shannon bound (3) and $E_{de}(S^{n-1}; R, A)$.

6 Other problems

This section overviews some other combinatorial problems in which Theorem 1 leads to new results. Let $C \in H^n$ be a linear code (a linear subspace of the \mathbb{F}_2 -linear space). Then $d(C)$ equals the minimum

Hamming weight of a nonzero code vector (the Hamming weight is the norm corresponding to the Hamming distance $\partial(\cdot, \cdot)$). Let $\dim C = k$ and let G be the $(k \times n)$ matrix whose rows form a basis of C (we assume that G has no all-zero columns). Columns of G can be viewed a multiset X of points in the $(k - 1)$ -dimensional projective space $\mathbb{P}(H^k)$; then clearly

$$d(C) = n - \max_{\mathcal{H}} |\{X \cap \mathcal{H}\}|$$

where the maximum is taken over all hyperplanes in $\mathbb{P}(H^k)$. Likewise if $\text{codim } \mathcal{H} = r \geq 2$, the corresponding value is called the r th (higher) weight of C , denoted $d_r(C)$. Properties of higher weights were a subject of intensive study in the 1990s [42]. One of the problems that present interest is finding $\max\{|C| : d_r(C) = d\}$ for a given $r \geq 2$. Best known asymptotic upper bounds on this quantity were proved in [5], an essential ingredient of the proof being Theorem 3 and related results.

Variations of the polynomial method proved to be efficient for deriving new upper bounds on the maximum size of list-decodable codes [3], on the covering radius of codes with a given dual distance [6], and on the minimum distance of doubly-even self-dual codes [27].

The polynomial method also proved useful in the study of quantum information transmission. It was applied in [7] to derive upper bounds on the size of quantum codes. In [2] the concept of error detection was extended to quantum codes. It turned out that the probability of undetected error can be expressed via the weight enumerators of quantum codes (the Shor-Laflamme enumerators [40]) in a way similar to $P_{ue}(H^n; C, p)$. It is shown in [2] that there exist quantum codes for which the probability of undetected error is an exponentially falling function of the code length n . Furthermore, a version of the polynomial method developed in [7], [2] leads to upper bounds on this exponent which are tight in a certain region of code rates.

References

- [1] A. Ashikhmin and A. Barg, *Binomial moments of the distance distribution: Bounds and applications*, IEEE Trans. Inform. Theory **45** (1999), no. 2, 438–452.
- [2] A. Ashikhmin, A. Barg, E. Knill, and S. Litsyn, *Quantum error detection, I-II*, IEEE Trans. Inform. Theory, submitted. Available as lanl e-prints quant-ph/9906126 and quant-ph/99-6131.
- [3] A. Ashikhmin, A. Barg, and S. Litsyn, *A new upper bound on codes decodable into size-2 lists*, Numbers, Information and Complexity, Kluwer Publishers.
- [4] ———, *A new upper bound on the reliability function of the Gaussian channel*, IEEE Trans. Inform. Theory, submitted.
- [5] ———, *New bounds on generalized weights*, IEEE Trans. Inform. Theory **45** (1999), no. 4, 1258–1263.
- [6] A. Ashikhmin, I. Honkala, S. Litsyn, and T. Laihonon, *On relations between covering radius and dual distance*, IEEE Trans. Inform. Theory **45**, no. 6, 1808–1816.
- [7] A. Ashikhmin and S. Litsyn, *Upper bounds on the size of quantum codes*, IEEE Trans. Inform. Theory **45** (1999), no. 4, 1206–1215.
- [8] E. Bannai and T. Ito, *Algebraic combinatorics I. Association schemes*, Benjamin/Cummings, London e. a., 1984.
- [9] A. Barg, *On polynomial invariants of codes, matroids, and finite interaction models*, preprint.
- [10] A. Barg and A. Ashikhmin, *Binomial moments of the distance distribution and the probability of undetected error*, Designs, Codes and Cryptography **16** (1999), no. 2, 103–116.
- [11] L. A. Bassalygo, *New upper bounds for error-correcting codes*, Problemy Peredachi Informacii **1** (1965), no. 4, 41–44.
- [12] L.-C. Chen and M. E. H. Ismail, *On asymptotics of Jacobi polynomials*, SIAM J. Math. Anal. **22** (1991), no. 5, 1442–1449.
- [13] P. Delsarte, *Bounds for unrestricted codes, by linear programming*, Philips Res. Rep. **27** (1972), 272–289.

- [14] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Research Repts Suppl. **10** (1973), 1–97.
- [15] P. D. Dragnev and E. B. Saff, *Constrained energy problems with applications to orthogonal polynomials of a discrete variable*, J. Anal. Math. **72** (1997), 223–259.
- [16] Á. Elbert, A. Laforgia, and Lucia G. Rodonò, *On the zeros of Jacobi polynomials*, Acta Math. Hungar. **64** (1994), no. 4, 351–359.
- [17] P. Elias, *Coding for two noisy channels*, Information Theory. Third London Symposium (C. Cherry, ed.), Academic Press, New York, 1956, pp. 61–74.
- [18] A. Feinstein, *A new basic theorem of information theory*, IRE Trans. Inform. Theory **PGIT-4** (1954), 2–22.
- [19] R. G. Gallager, *A simple derivation of the coding theorem and some applications*, IEEE Trans. Inform. Theory **11** (1965), 3–18.
- [20] ———, *Information theory and reliable communication*, John Wiley & Sons, New York e.a., 1968.
- [21] W. Gawronski and B. Sawyer, *Strong asymptotics and the limit distribution of the zeros of Jacobi polynomials $P_n^{(\alpha n + \alpha, \beta n + \beta)}$* , Progress in approximation theory, Academic Press, Boston, MA, 1991, pp. 379–404.
- [22] E. N. Gilbert, *A comparison of signalling alphabets*, Bell Syst. Techn. Journ. **31** (1952), 504–522.
- [23] C. D. Godsil, *Algebraic combinatorics*, Chapman & Hall, New York, 1993.
- [24] G. Kabatyansky and V. I. Levenshtein, *Bounds for packings on the sphere and in the space*, Problemy Peredachi Informatsii **14** (1978), no. 1, 3–25.
- [25] G. Kalai and N. Linial, *On the distance distribution of codes*, IEEE Trans. Inform. Theory **41** (1995), no. 5, 1467–1472.
- [26] A. I. Khinchin, *Mathematical foundations of information theory*, Dover Publications, Inc., New York, N. Y., 1957.
- [27] I. Krasikov and S. Litsyn, *An improved bound on the minimum distance of doubly-even codes*, IEEE Trans. Inform. Theory, to appear.
- [28] A. B. J. Kuijlaars and W. van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory **99** (1999), 167–197.
- [29] V. I. Levenshtein, *Bounds for packings of metric spaces and some of their applications*, Problemy Kibernet. (1983), no. 40, 43–110 (In Russian).
- [30] ———, *A straight-line bound for the exponent of the probability of the nondetection of an error*, Problemy Peredachi Informatsii **25** (1989), no. 1, 33–37.
- [31] ———, *Universal bounds for codes and designs*, Handbook of Coding Theory (V. Pless and W. C. Huffman, eds.), vol. 1, Elsevier Science, Amsterdam, 1998, pp. 499–648.
- [32] S. Litsyn, *New upper bounds on error exponents*, IEEE Trans. Inform. Theory **45** (1999), no. 2, 385–398.
- [33] R. J. McEliece and J. K. Omura, *An improved upper bound on the block coding error exponent for binary-input discrete memoryless channels*, IEEE Trans. Inform. Theory **23** (1977), no. 5, 611–613.
- [34] R. J. McEliece, E. R. Rodemich, H. Rumsey, and L. R. Welch, *New upper bound on the rate of a code via the Delsarte-MacWilliams inequalities*, IEEE Trans. Inform. Theory **23** (1977), no. 2, 157–166.
- [35] D. S. Moak, E. B. Saff, and R. S. Varga, *On the zeros of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}(x)$* , Trans. Amer. Math. Soc. **249** (1979), no. 1, 159–162.
- [36] E. R. Rodemich, *An inequality in coding theory*, AMS Abstract 773-05-39, presented at the Amer. Math. Soc. Meeting, San-Antonio, TX, Jan. 1980.
- [37] C. E. Shannon, *A mathematical theory of communication*, Bell Syst. Techn. Journ. **27** (1948), 379–423, 623–656.
- [38] ———, *Probability of error for optimal codes in a Gaussian channel*, Bell Syst. Techn. Journ. **38** (1959), no. 3, 611–656.
- [39] C. E. Shannon, R. G. Gallager, and E. R. Berlekamp, *Lower bounds to error probability for codes on discrete memoryless channels, II*, Information and Control **10** (1967), 522–552.
- [40] P. W. Shor and R. Laflamme, *Quantum analog of the MacWilliams identities in classical coding theory*, Phys. Rev. Lett **78** (1997), 1600–1602.
- [41] G. Szegő, *Orthogonal polynomials*, Colloquium Publications, vol. 23, AMS, Providence, RI, 1975.
- [42] M. A. Tsfasman and S. G. Vlăduț, *Geometric approach to higher weights*, IEEE Trans. Inform. Theory **41** (1995), 1564–1588.
- [43] R. R. Varshamov, *Estimate of the number of signals in error correcting codes*, Dokl. Akad. Nauk SSSR **117** (1957), 739–741.